ON 4-DIMENSIONAL, CONFORMALLY FLAT, ALMOST ε-KÄHLERIAN MANIFOLDS

KARINA OLSZAK AND ZBIGNIEW OLSZAK

Dedicated to the memory of Jerzy Julian Konderak

Abstract. The local structure of 4-dimensional, conformally flat, almost ε-Kählerian (i.e., almost pseudo-Kählerian and almost para-Kählerian) manifolds is characterized with the help of left-regular and right-regular paraquaternionic functions. Examples of such structures are discussed.

1. Introduction

Blair [1] has applied the quaternionic analysis to prove that almost Kählerian manifolds of dimension 4 cannot be of constant non-zero curvature. Recently, Królikowski [9, 10, 11] has developed the ideas of applications of quaternionic analysis during investigations of conformally flat almost Kählerian structures.

In the presented paper, applying the theory of regular paraquaternionic functions introduced by Pogoruy and Rodríguez-Dagnino [15], we propose paraquaternionic analogy of these ideas applied to conformally flat almost pseudo-Kählerian as well as almost para-Kählerian manifolds.

2. Regular paraquaternionic functions

Let $\mathbb{A}$ be the paraquaternion algebra over the real field $\mathbb{R}$ (which is also called the split-quaternion or coquaternion algebra); cf. [4, 13, 15, 16], etc.

Elements of $\mathbb{A}$ are of the form $\mathtt{x} = x^0 + \sum_\alpha x^\alpha i_\alpha$, where $\sum_\alpha$ denotes the sum with respect to $\alpha = 1, 2, 3$, and $x^0, x^\alpha \in \mathbb{R}$, and $1, i_1, i_2, i_3$ is a basis of $\mathbb{A}$ such that

$$i_1^2 = -1,\ i_2^2 = i_3^2 = 1,\ i_1 i_2 = -i_2 i_1 = i_3, i_2 i_3 = -i_3 i_2 = -i_1,\ i_1 i_3 = -i_3 i_1 = -i_2.$$

Thus, the multiplication rule of paraquaternions is as follows

$$\begin{aligned}\mathtt{xy} &= \Big(x^0 + \sum\nolimits_\alpha x^\alpha i_\alpha\Big)\Big(y^0 + \sum\nolimits_\alpha y^\alpha i_\alpha\Big)\\ &= x^0y^0 - x^1y^1 + x^2y^2 + x^3y^3 + (x^0y^1 + x^1y^0 - x^2y^3 + x^3y^2)i_1\\ &\quad + (x^0y^2 - x^1y^3 + x^2y^0 + x^3y^1)i_2 + (x^0y^3 + x^1y^2 - x^2y^1 + x^3y^0)i_3.\end{aligned}$$

$\mathbb{A}$ is an associative and non-commutative abstract algebra, which contains zero divisors, nilpotent elements, and nontrivial idempotents. The paraquaternionic algebra is isomorphic with the Clifford algebras $\mathcal{C}l(1,1) \cong \mathcal{C}l(0,2)$. The paraquaternions are most familiar through their isomorphism with 2×2 real matrices.

Date: October 18, 2011.

2000 *Mathematics Subject Classification.* Primary 53C55; secondary 53C25.

Key words and phrases. Paraquaternion, regular paraquaternionic function, conformally flat space, almost para-Kählerian manifold, almost pseudo-Kählerian manifold.

For a paraquaternion $\mathtt{x}$, define the real part and the imaginary part of $\mathtt{x}$ by

$$\mathfrak{Re}(\mathtt{x}) = x^0, \quad \mathfrak{Im}(\mathtt{x}) = \sum\nolimits_\alpha x^\alpha i_\alpha,$$

respectively. The conjugate of $\mathtt{x}$ is defined by $\overline{\mathtt{x}} = \mathfrak{Re}\,\mathtt{x} - \mathfrak{Im}(\mathtt{x})$, and the norm by

$$\sqrt{\mathtt{x}\overline{\mathtt{x}}} = \sqrt{(x^0)^2 + (x^1)^2 - (x^2)^2 - (x^3)^2}.$$

When $\mathtt{x}\overline{\mathtt{x}} < 0$, then the value of the norm is a complex number from the upper half of the complex plane.

The notions of left-regular and right-regular paraquaternionic functions was introduced by Pogoruy and Rodríguez-Dagnino in [15] by analogy to the notion of regular quaternionic functions. For quaternionic analysis, we refer e.g. [3, 7, 18], etc.

In the sequel, by $\mathcal{F}$ we will denote the set of all real-differentiable functions $f\colon U \to \mathbb{A}$, where U is a domain in $\mathbb{R}^4$.

Consider the following two Cauchy-Fueter type operators D_L and D_R defined on functions $f \in \mathcal{F}$ by the formulas (we apply denotations different from those in [15])

$$D_L f = \partial_0 f + \sum\nolimits_\alpha i_\alpha(\partial_\alpha f), \quad D_R f = \partial_0 f + \sum\nolimits_\alpha (\partial_\alpha f) i_\alpha,$$

where it is supposed that $\partial_j = \partial/\partial_j$. Writing f in the following form $f = f^0 + \sum_\alpha f^\alpha i_\alpha$, where $f^0, f^\alpha\colon U \to \mathbb{R}$, by straighforward computation, we claim for $D_L f$,

$$\begin{aligned} D_L f \;=\; & \partial_0 f^0 - \partial_1 f^1 + \partial_2 f^2 + \partial_3 f^3 + \big(\partial_1 f^0 + \partial_0 f^1 + \partial_3 f^2 - \partial_2 f^3\big) i_1 \\ & + \big(\partial_2 f^0 + \partial_3 f^1 + \partial_0 f^2 - \partial_1 f^3\big) i_2 + \big(\partial_3 f^0 - \partial_2 f^1 + \partial_1 f^2 + \partial_0 f^3\big) i_3, \end{aligned} \tag{1}$$

and for $D_R f$,

$$\begin{aligned} D_R f \;=\; & \partial_0 f^0 - \partial_1 f^1 + \partial_2 f^2 + \partial_3 f^3 + \big(\partial_1 f^0 + \partial_0 f^1 - \partial_3 f^2 + \partial_2 f^3\big) i_1 \\ & + \big(\partial_2 f^0 - \partial_3 f^1 + \partial_0 f^2 + \partial_1 f^3\big) i_2 + \big(\partial_3 f^0 + \partial_2 f^1 - \partial_1 f^2 + \partial_0 f^3\big) i_3. \end{aligned} \tag{2}$$

From (1) and (2), one notes that it always holds $\mathfrak{Re}(D_L f) = \mathfrak{Re}(D_R f)$.

The operators D_L and D_R are completely different. In the first lemma, we describe the paraquaternionic functions for which they give the same efects.

Lemma 1. *For a paraquaternionic function $f = f^0 + \sum_\alpha f^\alpha i_\alpha$, the condition $D_L f = D_R f$ holds if and only if $f^\alpha = \partial_\alpha F$ locally (i.e., in a neighborhood of an arbitrary point), where F is a certain paraquaternionic function.*

Proof. Using (1) and (2), we find

$$D_L f - D_R f = 2\big(\partial_3 f^2 - \partial_2 f^3\big) i_1 + 2\big(\partial_3 f^1 - \partial_1 f^3\big) i_2 + 2\big(\partial_1 f^2 - \partial_2 f^1\big) i_3.$$

Hence, the condition $D_L f = D_R f$ holds if and only if

$$\partial_3 f^2 - \partial_2 f^3 = 0, \quad \partial_3 f^1 - \partial_1 f^3 = 0, \quad \partial_1 f^2 - \partial_2 f^1 = 0.$$

When fixing x^0, the above condtion says that the 1-form $\omega = f^1 dx^1 + f^2 dx^2 + f^3 dx^3$ is closed. By the famous Poincare lemma, the form ω is locally a boundary form, say $\omega = dF$ for a certain paraquaternionic function F. This completes the proof. $\square$

Lemma 2. *The operators D_L and D_R commute, that is, for any $f \in \mathcal{F}$,*

$$D_L D_R f - D_R D_L f = 0. \tag{3}$$

Proof. Since the partial derivatives of functions f^i, $0 \leqslant i \leqslant 3$, commute and the paraquaternionic multiplication is associative, the conclusion follows obviously. $\square$

Let $f \in \mathcal{F}$. f is called left-regular if it satisfies the Cauchy-Riemann-Feuter type equation $D_L f = 0$. And similarly, f is called right-regular if it satisfies the Cauchy-Riemann-Feuter type equation $D_R f = 0$.

Having (1) and (2), we see that f is left-regular if and only if

$$\begin{cases} \partial_0 f^0 - \partial_1 f^1 + \partial_2 f^2 + \partial_3 f^3 = 0, \\ \partial_1 f^0 + \partial_0 f^1 + \partial_3 f^2 - \partial_2 f^3 = 0, \\ \partial_2 f^0 + \partial_3 f^1 + \partial_0 f^2 - \partial_1 f^3 = 0, \\ \partial_3 f^0 - \partial_2 f^1 + \partial_1 f^2 + \partial_0 f^3 = 0, \end{cases} \tag{4}$$

and f is right-regular if and only if

$$\begin{cases} \partial_0 f^0 - \partial_1 f^1 + \partial_2 f^2 + \partial_3 f^3 = 0, \\ \partial_1 f^0 + \partial_0 f^1 - \partial_3 f^2 + \partial_2 f^3 = 0, \\ \partial_2 f^0 - \partial_3 f^1 + \partial_0 f^2 + \partial_1 f^3 = 0, \\ \partial_3 f^0 + \partial_2 f^1 - \partial_1 f^2 + \partial_0 f^3 = 0. \end{cases} \tag{5}$$

Lemma 3. *(a) If f is a left-regular paraquaternionic function, then $h = D_R f$ is also left-regular paraquaternionic and $\mathfrak{Re}(h) = 0$. (b) If f is a right-regular paraquaternionic function, then $h = D_L f$ is also right-regular paraquaternionic and $\mathfrak{Re}(h) = 0$.*

Proof. (a) Since $D_L f = 0$, by (3), we have $D_L h = D_L D_R f = D_R D_L f = 0$. Moreover, using formula (2), we find

$$\mathfrak{Re}(h) = \mathfrak{Re}(D_R f) = \partial_0 f^0 - \partial_1 f^1 + \partial_2 f^2 + \partial_3 f^3. \tag{6}$$

But, by (1), the right hand side of (6) is the same as $\mathfrak{Re}(D_L f)$, which is equal to 0 since $D_L f = 0$.

(b) Since $D_R f = 0$, by (3), we have $D_R h = D_R D_L f = D_L D_R f = 0$. Moreover, using formula (1), we find

$$\mathfrak{Re}(h) = \mathfrak{Re}(D_L f) = \partial_0 f^0 - \partial_1 f^1 + \partial_2 f^2 + \partial_3 f^3. \tag{7}$$

But, by (2), the right hand side of (7) is the same as $\mathfrak{Re}(D_R f)$, which is equal to 0 since $D_R f = 0$. $\square$

The simplest examples of left-regular and right-regular paraquaternionic functions are the so-called paraquaternionic Fueter polynomials

$$\zeta_\alpha(x) = x^\alpha - x^0 i_\alpha, \quad x \in \mathbb{R}^4.$$

From the result achieved in [15] it follows that a paraquaternionic function is left-regular on U if and only if it can be expanded, in a neighborhood of an arbitrary point $x_0 \in U$, into a series

$$f(x) = f(x_0) + \sum_{n=1}^{\infty} \Big(\sum_{\alpha_1, \alpha_2, \ldots, \alpha_n = 1}^{3} \zeta_{\alpha_1}(x - x_0) \zeta_{\alpha_2}(x - x_0) \cdots \zeta_{\alpha_n}(x - x_0) a_{\alpha_1 \alpha_2 \cdots \alpha_n} \Big),$$

where $a_{\alpha_1 \alpha_2 \cdots \alpha_n} \in \mathbb{A}$ for any $1 \leqslant \alpha_i \leqslant 3$, $1 \leqslant i \leqslant n$ and $n \in \mathbb{N}$.

Hence, we claim that the paraquaternionic functions of the form

$$f(x) = \sum_{k=1}^{n} \Big(\sum_{\alpha_1, \alpha_2, \ldots, \alpha_n = 1}^{3} \zeta_{\alpha_1}(x) \zeta_{\alpha_2}(x) \cdots \zeta_{\alpha_k}(x) a_{\alpha_1 \alpha_2 \cdots \alpha_k} \Big),$$

where $a_{\alpha_1 \alpha_2 \cdots \alpha_k} \in \mathbb{A}$, are left-regular.

In a similar way, we can claim tha the paraquaternionic functions of the form

$$f(x) = \sum_{k=1}^{n} \Big(\sum_{\alpha_1, \alpha_2, \dots, \alpha_n = 1}^{3} a_{\alpha_1 \alpha_2 \cdots \alpha_k} \zeta_{\alpha_1}(x) \zeta_{\alpha_2}(x) \cdots \zeta_{\alpha_k}(x) \Big),$$

where $a_{\alpha_1 \alpha_2 \cdots \alpha_k} \in \mathbb{A}$, are right-regular.

3. Conformal flatness of 4-dimensional almost ε-Kählerian manifolds

Let M a 4-dimensional, connected, differentiable manifold. Assume that M is endowed with a pseudo-Riemannian metric g of Norden signature $(--++)$ and a $(1,1)$-tensor field J such

$$J^2 = \varepsilon I, \quad g(JX, JY) = -\varepsilon g(X, Y)$$

for any $X, Y \in \mathfrak{X}(M)$, where ε is a constant equal to $+1$ or -1.

If $\varepsilon = -1$, then the manifold $M(J, g)$ is an almost pseudo-Hermitian or indefinite almost Hermitian (cf. [12, 14], etc.). If $\varepsilon = +1$, then the manifold $M(J, g)$ is an almost para-Hermitian (cf. [2, 5], etc.). For simplicity, we will say that $M(J, g)$ or just M is an almost ε-Hermitian manifold.

For an almost ε-Hermitian manifold $M(J, g)$, let Ω be the skew-symmetric $(0,2)$-tensor file defined by $g(X, JY) = \Omega(X, Y)$. Ω is an almost symplectic form on M and it is called the fundamental form of this manifold.

Let $M(J, g)$ be a 4-dimensional almost ε-Hermitian manifold. Assume additionally that the manifold $M(J, g)$ is (locally) conformally flat. This means that for any point $p \in M$ there exist an open neighborhood $U \subset M$ of the point p and a positive function $h \colon U \to \mathbb{R}$ such that $g = hG$, where G is a flat pseudo-Riemannian metric. We restrict our consideration to the subset U. Without lose of generality, we assume that U is a coordinate neighborhood with coordinates (x^0, x^1, x^2, x^3) (it will be useful to number the indices from 0 to 3) and the metric G has components

$$G_{11} = G_{22} = -1, \quad G_{33} = G_{44} = 1, \quad \text{and } G_{ij} = 0 \text{ otherwise.} \tag{8}$$

Thus, the metric g has the following components

$$g_{11} = g_{22} = -h, \quad g_{33} = g_{44} = h, \quad \text{and } g_{ij} = 0 \text{ otherwise.}$$

Let J^i_j be the components of the tensor field J. Now, we compute the components J^i_j having the equalities

$$\sum_s J^i_s J^s_j = \varepsilon \delta^i_j, \quad \sum_s g_{is} J^s_j + \sum_s g_{js} J^s_i = 0.$$

By some standard computations, we obtain the following two possibilities for the matrix $[J^i_j]$,

$$[J^i_j] = \begin{bmatrix} 0 & a & b & c \\ -a & 0 & -c & b \\ b & -c & 0 & -a \\ c & b & a & 0 \end{bmatrix}, \tag{9}$$

or

$$[J^i_j] = \begin{bmatrix} 0 & a & b & c \\ -a & 0 & c & -b \\ b & c & 0 & a \\ c & -b & -a & 0 \end{bmatrix}, \tag{10}$$

a, b, c being functions such that $a^2 - b^2 - c^2 = -\varepsilon$. Consequently, the matrices of the components Ω_{ij} of the fundamental form are of the form

$$[\Omega_{ij}] = \begin{bmatrix} 0 & -f^1 & -f^2 & -f^3 \\ f^1 & 0 & f^3 & -f^2 \\ f^2 & -f^3 & 0 & -f^1 \\ f^3 & f^2 & f^1 & 0 \end{bmatrix}, \tag{11}$$

or

$$[\Omega_{ij}] = \begin{bmatrix} 0 & -f^1 & -f^2 & -f^3 \\ f^1 & 0 & -f^3 & f^2 \\ f^2 & f^3 & 0 & f^1 \\ f^3 & -f^2 & -f^1 & 0 \end{bmatrix}, \tag{12}$$

where we supposed $f^1 = ha$, $f^2 = hb$ and $f^3 = hc$. The functions f^1, f^2 and f^3 satisfy the condition $(f^1)^2 - (f^2)^2 - (f^3)^2 = -\varepsilon h^2$.

The coordinate neighborhood $(U, (x^0, x^1, x^2, x^3))$ constructed in the above will be said to be the canonical coordinate neighborhood of a conformally flat 4-dimensional almost ε-Hermitian.

Assume that for an almost ε-Hermitian manifold $M(J, g)$, the fundamental form Ω is closed ($d\Omega = 0$, that is, Ω is a symplectic form). If $\varepsilon = -1$, then the manifold $M(J, g)$ is called almost pseudo-Kählerian or indefinite almost Kählerian (cf. e.g. [6, 17]). If $\varepsilon = +1$, then the manifold $M(J, g)$ is called almost para-Kählerian (cf. e.g. [8]). For simplicity, we will say that $M(J, g)$ or just M is an almost ε-Kählerian manifold.

Theorem 1. *A 4-dimensional, conformally flat, almost ε-Hermitian manifold $M(J, g)$ is almost ε-Kählerian if and only if for any point $p \in M$ there is a canonical coordinate neighborhood $(U, (x^0, x^1, x^2, x^3))$, $p \in U$, on which there exist functions f^1, f^2, f^3 and a positive function h such that*
(i) the functions f^1, f^2, f^3, h are related by the condition

$$(f^1)^2 - (f^2)^2 - (f^3)^2 = -\varepsilon h^2, \tag{13}$$

and the paraquaternionic function $f = f_1 i_1 + f^2 i_2 + f^3 i_3$ is left-regular or right-regular;
(ii) $g = hG$, G being the standard flat Norden type metric (cf. (8));
(iii) if f is left-regular, then J is given by (9), and if f is right-regular, then J is give by (10), where in the both cases, a, b, c are the functions given by

$$a = \frac{f^1}{h}, \quad b = \frac{f^2}{h}, \quad c = \frac{f^3}{h}.$$

Proof. Let $M(J, g)$ be a 4-dimensional, conformally flat, almost ε-Hermitian manifold. In a canonical coordinate neighborhood, the components of the exterior derivative $d\Omega$ are of the form

$$(d\Omega)_{ijk} = \frac{1}{3}\big(\partial_i \Omega_{jk} + \partial_j \Omega_{ki} + \partial_k \Omega_{ij}\big). \tag{14}$$

Let us assume that $M(J,g)$ is almost ε-Kählerian. Since $d\Omega = 0$, from (14), we obtain

$$\partial_i \Omega_{jk} + \partial_j \Omega_{ki} + \partial_k \Omega_{ij} = 0. \tag{15}$$

In the case when Ω is of the form (11), the relations (15) can be written equivalently as the following system

$$\begin{cases} -\partial_1 f^1 + \partial_2 f^2 + \partial_3 f^3 = 0, \\ \quad \partial_0 f^1 + \partial_3 f^2 - \partial_2 f^3 = 0, \\ \quad \partial_3 f^1 + \partial_0 f^2 - \partial_1 f^3 = 0, \\ -\partial_2 f^1 + \partial_1 f^2 + \partial_0 f^3 = 0. \end{cases}$$

Comparing this system with (4) we claim that the paraquaternionic function $f = f^1 i_1 + f^2 i_2 + f^3 i_3$ is left-regular.

In the case when Ω is of the form (12), the relations (15) can be written equivalently as the following system

$$\begin{cases} -\partial_1 f^1 + \partial_2 f^2 + \partial_3 f^3 = 0, \\ \quad \partial_0 f^1 - \partial_3 f^2 + \partial_2 f^3 = 0, \\ -\partial_3 f^1 + \partial_0 f^2 + \partial_1 f^3 = 0, \\ \quad \partial_2 f^1 - \partial_1 f^2 + \partial_0 f^3 = 0. \end{cases}$$

Comparing this system with (5), we claim that the paraquaternionic function $f = f^1 i_1 + f^2 i_2 + f^3 i_3$ is right-regular.

Recall that the function h deforming the metric g to a flat metric is related to the functions f^1, f^2, f^3 by (13).

The converse is clear. $\square$

In view of Theorem 1, to construct examples of 4-dimensional conformally flat almost ε-Kählerian structures it is necessary and sufficient to have left-regular and right-regular paraquaternionic functions $f = f^1 i_1 + f^2 i_2 + f^3 i_3$ for which $(f^1)^2 - (f^2)^2 - (f^3)^2 \neq 0$. Using the results from the previous section, one can find such functions. We are going to state only two concrete examples of such functions. Namely,

(a) the function

$$\begin{aligned} f(x) &= (\zeta_1(x) + \zeta_2 x) + \zeta_3(x))(- i_2 + i_3) \\ &= 2x^0 i_1 + (x^0 - x^1 - x^2 - x^3) i_2 + (x^0 + x^1 + x^2 + x^3) i_3, \end{aligned}$$

which is left-regular and realizes the desired conditions on any domain in $\mathbb{R}^4$ on which $|x^0| \neq |x^1 + x^2 + x^3|$; and similarly

(b) the function

$$\begin{aligned} f(x) &= (i_2 - i_3)(\zeta_1(x) + \zeta_2 x) + \zeta_3(x)) \\ &= 2x^0 i_1 + (x^0 + x^1 + x^2 + x^3) i_2 + (x^0 - x^1 - x^2 - x^3) i_3, \end{aligned}$$

which is right-regular and realizes the desired conditions on any domain in $\mathbb{R}^4$ on which $|x^0| \neq |x^1 + x^2 + x^3|$. Having these functions, one can write down the almost ε-Kählerian structures explicitly.

References

[1] D. E. Blair, *Nonexistence of 4-dimensional almost Kaehler manifolds of constant curvature*, Proc. Amer. Math. Soc. **110** (1990), 1033-1039.

[2] A. Borowiec, M. Ferraris, M. Francaviglia, I. Volovich, *Almost-complex and almost-product Einstein manifolds from a variational principle*, J. Math. Phys. **40** (1999), 3446-3464.

[3] J. Bureš, R. Lávička and V. Souček, *Elements of quaternionic analysis and random transforms*, Textos de Matemática. Série B 42, Coimbra: Universidade de Coimbra, Departamento de Matemática. vii, (2009).

[4] J. Cockle, *On systems of algebra involving more than one imaginary*, Philos. Mag., III. Ser. **35** (1849), 434-435.

[5] V. Cruceanu, P. Fortuny and P. M. Gadea, *A survey on paracomplex geometry*, Rocky Mountain J. Math. **26** (1996), 83–115.

[6] J. Davidov, J. C. Díaz-Ramos, E. García-Río, Y. Matsushita, O. Muškarov and R. Vázquez-Lorenzo, *Almost Kähler Walker 4-manifolds*, J. Geom. Phys. **57** (2007), 1075-1088.

[7] R. Fueter, *Analitische Functionen einer Quaternionen variablen*, Comment. Math. Helv. **4** (1932), 9-20.

[8] S. Ivanov and S. Zamkovoy, *ParaHermitian and paraquaternionic manifolds*, Diff. Geom. Appl. **23** (2005), 205-234.

[9] W. Królikowski, *On Clifford-type structures*, Diss. Math. (Rozprawy Mat.) **440** (2006), Inst. Math., Polish Acad. Sci., Warsaw.

[10] W. Królikowski, *On 4-dimensional locally conformally flat almost Kähler manifolds*, Arch. Math. Brno **42** (2006), 215-223.

[11] W. Królikowski, *Quaternionic condition for the existence of 4-dimensional locally conformally flat almost Kähler manifolds*, Demonstr. Math. **39** (2006), 195-202.

[12] Y. Matsushita, *Pseudo-Chern classes of an almost pseudo-Hermitian manifold*, Trans. Amer. Math. Soc. **301** (1987), 665-677.

[13] S. Marchiafava, *Twistorial maps between (para)quaternionic projective spaces*, Bull. Math. Soc. Sci. Math. Roumanie **52**(**100**) (2009), 321-332.

[14] R. K. Nagaich, *Constancy of holomorphic sectional curvature in indefinite almost Hermitian manifolds*, Kodai Math. J. **16** (1993), 327-331.

[15] A. A. Pogoruy and R. M. Rodríguez-Dagnino, *Some algebraic and analytical properties of coquaternion algebra*, Adv. Appl. Clifford Alg. **20** (2010), 79-84.

[16] B. Rosenfeld, *Geometry of Lie groups*, Mathematics and its Applications Vol. 393, Kluwer Academic Publishers, Gordrecht, 1977.

[17] K. Sekigawa and A. Yamada, *Compact indefinite almost Kähler Einstein manifolds*, Geom. Dedicata **132** (2008), 65-79.

[18] A. Sudbury, *Quaternionic analysis*, Math. Proc. Camb. Philos. Soc. **85** (1979), 199-225.

Institute of Mathematics and Computer Science, Wrocław University of Technology, Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland

E-mail address: `karina.olszak@pwr.wroc.pl, zbigniew.olszak@pwr.wroc.pl`